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ON THE OPTIMAL CONTROL OF A SORPTION PROCESS*

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A system of equations of sorption is examined. The velocity of movement of the matter is a control parameter. The existence of an optimal control is proved for certain forms of the cost function. Necessary optimality conditions are obtained and proved.

In $Q = (0, T) \times (0, l)$ we examine the system

$$R_t + vR_x = W - R,\ W_t = R - W \tag{0.1}$$
$$R(x, 0) = W(x, 0) = \varphi(x),\ R(0, t) = 0$$

$v = v(t) \in V = \{v : 0 < \alpha_0 \leqslant \alpha(t) \leqslant v \leqslant \beta(t) \leqslant \beta_0$ almost everywhere on $(0, T)\}$. Physically R and W are the concentrations of the unsorbed and sorbed phases of the matter, respectively, v, the velocity of motion of the exhaust gas, of the wash water, etc., is a control parameter, V is the set of admissible controls ($(\alpha(t), \beta(t)$ are bounded measurable functions, α_0, β_0 are constants). We are required to find a control $v \in V$ which minimizes the functional

$$J(v) = \int_0^T [(v(t) R(l, t) - z(t))^2 + 2\nu v(t)]\, dt \tag{0.2}$$

where $\nu > 0$ is a constant. For soil desalinazation problems /1/, for example, the minimization of (0.2) signifies the approximation of the velocity of salt removal from layer $(0, l)$ to a prescribed velocity $z(t)$ with a minimal amount of expenditure of water in doing so. In the paper we prove the existence of the optimal control for a certain class of initial conditions and modes. We derive necessary conditions which the optimal control must satisfy. Functional (0.2) is examined only for the example; the proof carries over without difficulty to the case of functionals of other types.

1. Existence of the optimal control.

The following notation is used for the functional spaces: $L^p(Q)$ is the space of functions integrable to the pth degree in Q; $L^\infty(Q)$ is the space of functions essentially bounded in Lebesgue measure in Q; $H^1(Q) = W^{1,2}(Q) = \{u : u, \partial u/\partial x, \partial u/\partial t \in L^2(Q)\}$ is a Sobolev space; $L^p(0, T; X) = \{f : f$ is the measurable mapping

$$(0, T) \to X:\ \left(\int_0^T \|f(t)\|_X{}^p\, dt\right)^{1/p} < +\infty \text{ for } 1 \leqslant p < +\infty \text{ and } \operatorname*{supess}_{t\in(0,T)} \|f(t)\|_X < +\infty \text{ for } p = +\infty\};\ D'(Q)$$

is the space of distribution in Q.

Theorem. Let $\varphi(x) \in H^1(0, l) \cap L^\infty(0, l)$, $z(t) \in L^2(0, T)$. Under these conditions a solution of problem (0.1) exists and is unique and an optimal control $v \in V$ minimizing functional (0.2) exists.

Proof. The existence and uniqueness of the solution of problem (0.1) can be proved analogously to the solution of the Carleman problem in /2/. Here it can be proved that for $v \in V$ the norms of the solution can be estimated in terms of the norm of φ with the use of only the constants α_0, β_0. Under the theorem's conditions the solutions of the system

$$R,\ W \in L^\infty(0, T);\ H^1(0, l) \cap L^\infty(0, l))$$

and, as follows from (0.1), $R, W \in H^1(Q)$. By the same token, for the functions R, W we can determine their traces on the boundary Γ of domain Q, belonging to $L^2(\Gamma)$ /3/, and functional (0.2) makes sense. Functioanl $J(v)$ is bounded from below. Consequently, its inf on V exists. Let $\{v_n\}$ be a minimizing sequence, $\{R_n\}$, $\{W_n\}$ be the corresponding solutions of (0.1). By virtue of the boundedness of $\{v_n\}$ in $L^\infty(0, T)$ $(v_n \in V)$ and of $\{R_n\}, \{W_n\}$ in $L^\infty(0, T; H^1(0, l) \cap L^\infty(0, l))$ there exist subsequences (denoted the same) such that $\{v_n\} \to v$ *-weakly in $L^\infty(0, T)$, $\{R_n\} \to R$, $\{W_n\} \to W$ *weakly in $L^\infty(0, T;\ H^1(0, l) \cap L^\infty(0, l))$. Here $v \in V \in L^\infty(0, T)$, $R, W \in L^\infty(0, T;\ H^1(0, l) \cap L^\infty(0, l))$ by virtue of the *-weak closure of balls in these spaces. But since $R_n, W_n \in H^1(Q)$ and the inclusion

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$H^1(Q) \to L^2(Q)$ is compact, there exist subsequences $\{R_n\}, \{W_n\}: \{R_n\} \to R, \{W_n\} \to W$ strongly in $L^2(Q)$ and almost everywhere. For these subsequences, in (0.1) we can perform the limit passage to $D'(Q)$. By the same token, we have shown that v is the optimal control.

2. Necessary conditions for optimality of control. The optimal control v is characterized by the fact that for any $u \in V$ and $0 \leqslant \theta \leqslant 1$ the inequality $dJ_\theta/d\theta|_{\theta=0} | \geqslant 0$; $J_\theta = J(v_\theta)$, $v_\theta = v + \theta(u - v)$, is valid /3/. To compute the functional's derivative we consider (0.1) with v and v_θ (solutions R_θ, W_θ). In appropriate fashion we subtact one from the other. We obtain the problem

$$R_{*t} + vR_{*x} = W_* - R_* - (u - v) R_{\theta x}, \quad W_{*t} = R_* - W_* \tag{2.1}$$
$$R_*(x, 0) = W_*(x, 0) = R_*(0, t) = 0$$
$$R_* = (R_\theta - R)/\theta, \quad W_* = (W_\theta - W)/\theta$$

It is necessary to substantiate the possibility of a transition in (2.1) as $\theta \to 0$. From (2.1) we obtain the identity

$$\frac{1}{2}\frac{d}{dt}(|R_*|^2 + |W_*|^2) + \frac{1}{2}vR_*^2(l, t) + |R_* - W_*|^2 = \tag{2.2}$$
$$-\int_0^l (u - v) R_{\theta x} R_* \, dx, \quad |R|^2 = \int_0^l R^2(x, t)\, dx$$

Hence it follows that $R_*, W_* \in L^\infty(0, T; L^2(0, l))$ for any $0 \leqslant \theta \leqslant 1$. Having made the change of variables

$$x \to \zeta, t \to \tau : x = \zeta, \ x = \int_\tau^t v(t')\, dt'$$

from (2.1) we obtain that the functions $R_{*\zeta}, W_{*\tau}$ are bounded in $L^\infty(0, T; L^2(0, l))$. From Sect.1 it follows that the function $R_{\theta x}$ is bounded in $L^\infty(0, T; L^2(0, l))$. Consequently, there exists the subsequence $\{\theta_k\} \to 0 : \{R_{*k}\} \to y_1$, $\{W_{*k}\} \to y_2$ *-weakly in $L^\infty(0, T; L^2(0, l))$, $\{R_{*k\zeta}\} \to y_{1\zeta}$, $\{W_{*k\tau}\} \to y_{2\tau}$, $\{R_{\theta_k x}\} \to R_x$ *-weakly in $L^\infty(0, T; L^2(0, l))$. By the same token we can pass to the weak limit in (2.1). We get that here

$$y_{1t} + vy_{1x} = y_2 - y_1 - (u - v) R_x, \quad y_{2t} = y_1 - y_2 \tag{2.3}$$
$$y_1(x, 0) = y_2(x, 0) = y_1(0, t) = 0$$

Using the convergence properties of traces of functions with $x = l$, implying from (2.2), we obtain

$$\left.\frac{\partial J_\theta}{\partial \theta}\right|_{\theta=0} = 2\int_0^T [(vR(l, t) - z(t))(R(l, t)(u-v) + vy_1(l, t)) + \nu(u-v)]\, dt \geqslant 0$$

We consider one further auxiliary problem (conjugate with (0.1))

$$p_{1t} + vp_{1x} = p_1 - p_2, \quad p_{2t} = p_2 - p_1 \tag{2.4}$$
$$p_1(x, T) = p_2(x, T) = 0, \quad p_1(l, t) = v(t) R(l, t) - z(t)$$

According to the general theory of linear equations and hyperbolic systems /4/, there exists a unique generalized solution $p_1 p_2 \in L^2(Q)$ of problem (2.4). Hence if follows that

$$\left.\frac{\partial J_\theta}{\partial \theta}\right|_{\theta=0} = 2\int_0^T R^2(l, t)(v(t) - g(t))(u - v)\, dt \geqslant 0 \tag{2.5}$$

$$g(t) = \frac{1}{R^2(l, t)}\left(z(t) R(l, t) + \int_0^l R_x(x, t) p_1(x, t)\, dx - \nu\right)$$

for the optimal control v and for any $u \in V$. Consequently, the integrand in the first formula of (2.5) is nonnegative almost everywhere on $(0, T)$.

We introduce the function

$$\Phi(\alpha, \beta, g) = \begin{cases} \alpha(t), & g < \alpha(t) \\ \beta(t), & g > \beta(t) \\ g(t), & \alpha(t) \leqslant g \leqslant \beta(t) \end{cases}$$

we get that to solve the optimal control problem we need to solve the problem

$$
\begin{aligned}
&R_t + \Phi(\alpha, \beta, g) R_x = W - R, \quad W_t = R - W \\
&p_{1t} + \Phi(\alpha, \beta, g) p_{1x} = p_1 - p_2, \quad p_{2t} = p_2 - p_1 \\
&R(x, 0) = W(x, 0) = \varphi(x), \; R(0, t) = 0 \\
&p_1(x, T) = p_2(x, T) = 0, \; p_1(l, t) = \Phi(\alpha, \beta, g) \; R(l, t) - z(t)
\end{aligned} \tag{2.6}
$$

and the optimal control $v = \Phi(\alpha, \beta, g)$. In Sect.1 we proved the existence of the optimal control, viz., the solutions of problem (0.1) and (0.2). The solution of (2.6) corresponds to (0.1) and (0.2) and satisfies the necessary control optimality conditions. It is clear that the solution of (2.6) exists as well. The question of the uniqueness of the solutions of problems (2.6), (0.1) and (0.2) remains open by virtue of the nonlinearity of the mapping $v \to (R, W)$.

3. Other forms of the functionals. With slight changes the theory for functional (0.2) carries over to other functionals,for example

$$J(v) = \int_0^T [(vR(l, t) - z(t))^2 + \nu v^2]\, dt$$

$$J(v) = \int_0^T \left[\int_0^l (R + W - z)^2\, dx + \nu v^2\right] dt$$

$$J(v) = \int_0^l (R(x, T) + W(x, T) - z(x))^2\, dx + \nu \int_0^T v^2\, dt$$

For the last case the necessary condition appears thus:

$$g(t) = \frac{1}{\nu} \int_0^l R_x(x, t)\, p_1(x, t)\, dx$$

The function $\Phi(\alpha, \beta, g)$ is the same as in Sect.2. The final problem differs from (2.6) only in the conditions on p_1, p_2

$$p_1(l, t) = 0, \; p_1(x, T) = p_2(x, T) = R(x, T) + W(x, T) - z(x)$$

We remark that the optimal control problem can be solved by numerical methods /5/ without the use of problems of type (2.6).

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